

Generalised triangle groups of type $(3, 5, 2)$

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Abstract

If G is a group with a presentation of the form $\langle x, y | x^3 = y^5 = W(x, y)^2 = 1 \rangle$, then either G is virtually soluble or G contains a free subgroup of rank 2. This provides additional evidence in favour of a conjecture of Rosenberger.

1 Introduction

A *generalised triangle group* is a group G with a presentation of the form

$$\langle x, y | x^p = y^q = W(x, y)^r = 1 \rangle$$

where $p, q, r \geq 2$ are integers and $W(x, y)$ is a word of the form

$$W(x, y) = x^{\alpha(1)} y^{\underline{(1)}} \dots x^{\alpha(k)} y^{\underline{(k)}},$$

with $0 < \alpha(i) < p$ and $0 < \underline{(i)} < q$. We say that G is of *type* (p, q, r) . Without loss of generality, we assume that $p \leq q$.

A conjecture of Rosenberger [16] asserts that a Tits alternative holds for generalised triangle groups:

Conjecture A (Rosenberger) *Let G be a generalised triangle group. Then either G is soluble-by-finite or G contains a non-abelian free subgroup.*

In a recent article [12], we proved the Rosenberger Conjecture in the case $(p, q, r) = (3, 3, 2)$. In the present note we prove it in the case $(p, q, r) = (3, 5, 2)$. In conjunction with previously known results [8, 3, 11, 15, 4, 5, 1, 2, 6, 13, 14, 17] (see for example the survey [9] or the Introduction to [12] for details), this reduces the conjecture to the cases $(p, q, r) = (2, q, 2)$ for $q \in \{3, 4, 5\}$.

2 Preliminary results

Suppose that $X, Y \in SL(2, \mathbb{C})$ are matrices, and $W = W(X, Y)$ is a word in X, Y . Then the trace of W can be calculated as the value of a 3-variable polynomial, where the variables are the traces of X, Y and XY [10]. We can use this to find and analyse *essential representations* from G to $PSL(2, \mathbb{C})$. (A representation of G is *essential* if the images of $x, y, W(x, y)$ have orders p, q, r respectively.)

We can force the images x, y to have orders 3, 5 in $PSL(2, \mathbb{C})$ by mapping them to matrices $X, Y \in SL(2, \mathbb{C})$ of trace $2 \cos(\pi/3) = 1$ and $2 \cos(\pi/5) = (1 + \sqrt{5})/2$ respectively. Then the trace of $W(X, Y) \in SL(2, \mathbb{C})$ is given by a one-variable polynomial $\tau_W(l)$ of degree k , where l denotes the trace of XY . We obtain an essential representation by choosing l to be a root of τ_W (which forces the image of W to have order 2 in $PSL(2, \mathbb{C})$).

Lemma 2.1 *G contains a nonabelian free subgroup, unless the roots of $\tau_W(l)$ all belong to $\{0, 1, (1 + \sqrt{5})/2, (-1 + \sqrt{5})/2\}$.*

Proof. The image of an essential representation of G is generated by two elements of orders 3 and 5 respectively, and contains an element of order 2. With the exception of the finite group A_5 , any such subgroup of $PSL(2, \mathbb{C})$ contains a nonabelian free subgroup. The result follows, unless all essential representations $\rho : G \rightarrow PSL(2, \mathbb{C})$ have image isomorphic to A_5 .

Now let l be a root of $\tau_W(l)$ corresponding to an essential representation $\rho : G \rightarrow A_5$, where $\rho(x), \rho(y)$ are represented by matrices X, Y of traces $1, (1 + \sqrt{5})/2$ respectively. Then XY and XY^{-1} are matrices representing nontrivial elements of A_5 , which therefore have orders in $\{2, 3, 5\}$. Thus the traces of XY and XY^{-1} belong to $\{0, \pm 1, (\pm 1 \pm \sqrt{5})/2\}$. Moreover, these traces also satisfy the trace equation

$$tr(XY) + tr(XY^{-1}) = tr(X)tr(Y) = \frac{1 + \sqrt{5}}{2}.$$

From this, it follows that $l = tr(XY) \in \{0, 1, (\pm 1 + \sqrt{5})/2\}$, as claimed.

Lemma 2.2 *Let $p : \overline{K} \rightarrow K$ be a regular covering of connected 2-complexes with K finite, with covering transformation group abelian of torsion-free rank at least 2. Let F be a field. If*

$$H_2(\overline{K}, F) = 0 \neq H_1(\overline{K}, F),$$

then

$$\dim_F H_1(\overline{K}, F) = \infty.$$

Proof. Let $\{a, b\}$ be a basis for a free abelian subgroup A of the group of covering transformations of $p : \overline{K} \rightarrow K$, and let α be a cellular 1-cycle of \overline{K} over F that represents a non-zero element of $H_1(\overline{K}, F)$. If the $F[a]$ -submodule of $H_1(\overline{K}, F)$ generated by α is free, then $H_1(\overline{K}, F)$ is infinite-dimensional over F , as claimed. So we may assume that there is a cellular 2-chain β of \overline{K} with $d(\beta) = f(a)\alpha$ for some non-zero polynomial $f(a) \in F[a]$.

For similar reasons, we may also assume that $d(\gamma) = g(b)\alpha$ for some cellular 2-chain γ of \overline{K} and some non-zero polynomial $g(b) \in F[b]$.

Now $f(a)\gamma - g(b)\beta \in H_2(\overline{K}, F) = 0$. In other words $f(a)\gamma = g(b)\beta$ in the group $C_2(\overline{K}, F)$ of cellular 2-chains of \overline{K} , which is a free module over the unique factorisation domain $FA \cong F[a^{\pm 1}, b^{\pm 1}]$. Since $f(a), g(b)$ are coprime in $F[a^{\pm 1}, b^{\pm 1}]$, it follows that there is a 2-chain δ with $f(a)\delta = \beta$ and $g(b)\delta = \gamma$. Hence $f(a)(d(\delta) - \alpha) = d(\beta) - f(a)\alpha = 0$, in the group $C_1(\overline{K}, F)$ of cellular 1-chains of \overline{K} . But $C_1(\overline{K}, F)$ is also a free module over the domain $F[a^{\pm 1}, b^{\pm 1}]$, and $f(a) \neq 0$, so $d(\delta) = \alpha$, contradicting the hypothesis that α represents a non-zero element of $H_1(\overline{K}, F)$.

This contradiction completes the proof.

Lemma 2.3 *Let E be the set of midpoints of edges of a regular icosahedron $\mathcal{I} \subset \mathbb{R}^3$ centred at the origin, and let $M = \mathbb{Z}E$ its \mathbb{Z} -span in \mathbb{R}^3 . Let $V = \{1, a, b, c\} \subset \text{Isom}^+(\mathcal{I}) \subset SO(3)$ be the Klein 4-group, and let $C = \{1, c\} \subset V$. Then, regarding M as a $\mathbb{Z}V$ -module via the action of V by isometries of \mathcal{I} , we have the following.*

1. $M \cong \mathbb{Z}^6$ as an abelian group.
2. $H_0(C, M) = \mathbb{Z} \otimes_{\mathbb{Z}C} M \cong \mathbb{Z}_2^4 \oplus \mathbb{Z}^2$.
3. The induced action of V/C on $H_0(C, M)/(\text{torsion})$ is multiplication by -1 .

Proof. If e is the midpoint of the edge joining two vertices u, v of \mathcal{I} , then $e = (u + v)/2$. Thus E is contained in the \mathbb{Q} -span W of the set of vertices of \mathcal{I} . Since the vertices occur in 6 antipodal pairs, the \mathbb{Q} -span $\mathbb{Q}M$ of E has dimension at most 6 over \mathbb{Q} .

On the other hand, for any vertex v , $\sqrt{5} \cdot v$ is the sum of the 5 vertices adjacent to v in \mathcal{I} . Thus $\sqrt{5} \cdot v \in W$. It also follows that $\sqrt{5} \cdot e \in M$ for any $e \in E$: specifically, $(\sqrt{5} - 2) \cdot e$ is the sum of the midpoints of the eight edges of \mathcal{I} that share a vertex with the edge containing e . If $e_1, e_2, e_3 \in E$ are chosen to be linearly independent over \mathbb{R} – and hence over $\mathbb{Q}[\sqrt{5}]$ – then $e_1, e_2, e_3, \sqrt{5} \cdot e_1, \sqrt{5} \cdot e_2, \sqrt{5} \cdot e_3 \in M$ are linearly independent over \mathbb{Q} . Thus $\mathbb{Q}M = \mathbb{Q} \otimes_{\mathbb{Z}} M$ has dimension exactly 6 over \mathbb{Q} . Since $M \subset \mathbb{Q}M$ is torsion-free and finitely generated, it follows that $M \cong \mathbb{Z}^6$, as claimed.

If, in the above, we choose e_1, e_2, e_3 to lie on the axes of the rotations $a, b, c \in V$ respectively, then we obtain a decomposition

$$\mathbb{Q}M = \mathbb{Q}[\sqrt{5}]e_1 \oplus \mathbb{Q}[\sqrt{5}]e_2 \oplus \mathbb{Q}[\sqrt{5}]e_3$$

of $\mathbb{Q}M$ as a $\mathbb{Q}[\sqrt{5}]$ -vector space, with respect to which a, b, c act as the diagonal matrices $\text{diag}(1, -1, -1)$, $\text{diag}(-1, 1, -1)$ and $\text{diag}(-1, -1, 1)$ respectively. Let

$$M_+ := M \cap \mathbb{Q}[\sqrt{5}]e_3 \quad \text{and} \quad M_- := M \cap (\mathbb{Q}[\sqrt{5}]e_1 \oplus \mathbb{Q}[\sqrt{5}]e_2).$$

Then $M_- \cap M_+ = \{0\}$, while $e_1, e_2, \sqrt{5}e_1, \sqrt{5}e_2 \in M_-$ and $e_3, \sqrt{5}e_3 \in M_+$, so M_-, M_+ are free abelian of ranks 4 and 2 respectively.

Moreover, M/M_- is naturally embedded in the vector space $\mathbb{Q}M/\mathbb{Q}M_-$, so is also free abelian – necessarily of rank 2. Note that M_- is closed under the action of V on M . Under the induced action on M/M_- , each of a, b acts as the antipodal map, multiplication by -1 , and c acts as the identity.

Hence $(1 - c)M = 2M_-$, so

$$H_0(C, M) = M/(1 - c)M = M/2M_- \cong \mathbb{Z}_2^4 \oplus \mathbb{Z}^2,$$

as claimed.

Finally, the quotient of $H_0(C, M)$ by its torsion subgroup is naturally isomorphic to M/M_- , and the induced action of V/C on this quotient is via the antipodal map.

Lemma 2.4 *Let $G = \langle x, y | x^3 = y^5 = W(x, y)^2 = 1 \rangle$ and suppose that $(\lambda - \alpha)^2$ divides the trace polynomial $\tau_W(\lambda)$ of W , for some $\alpha \in \{0, 1, (1 + \sqrt{5})/2, (-1 + \sqrt{5})/2\}$. Let $\rho : G \rightarrow A_5$ be the natural epimorphism corresponding to the root α of $\tau_W(l)$. Let $C \subset A_5$ be a subgroup of order 2 and $V \subset A_5$ its centraliser of order 4. Then G has subgroups $N_1 \triangleleft N_2 \triangleleft \rho^{-1}(V)$ such that*

1. $\rho(N_2) = \{1\}$;
2. $\rho^{-1}(C)/N_2 \cong \mathbb{Z}^2$;
3. $\rho^{-1}(V)/N_2 \cong \mathbb{Z}^2 \rtimes_{(-1)} \mathbb{Z}_2$;
4. N_2/N_1 is a non-zero vector space over \mathbb{Z}_2 .

Proof. Let $\mathbb{L} = \mathbb{C}[\lambda]/\langle (\lambda - \alpha)^2 \rangle$, and choose matrices

$$X = \begin{pmatrix} e^{i\pi/3} & 0 \\ 1 & e^{-i\pi/3} \end{pmatrix}, \quad Y = \begin{pmatrix} e^{i\pi/5} & l - \alpha - 2\cos(8i\pi/15) \\ 0 & e^{-i\pi/5} \end{pmatrix} \in SL_2(\mathbb{L})$$

so that

$$\text{tr}(X) = 1, \quad \text{tr}(Y) = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \text{tr}(XY) = \lambda - \alpha.$$

Then X, Y determine a representation $\hat{\rho} : G \rightarrow PSL_2(\mathbb{L})$, since $\text{tr}(W(X, Y)) = \tau_W(l) = 0$ in \mathbb{L} . If $\phi : PSL_2(\mathbb{L}) \rightarrow PSL_2(\mathbb{C})$ is the natural epimorphism obtained by setting $\lambda = \alpha$, then the image of $\rho = \phi \circ \hat{\rho}$ is isomorphic to A_5 . Let K denote the kernel of ρ and let L denote the kernel of $\hat{\rho}$.

Clearly $G/K \cong A_5$. Now $K/L \cong \hat{\rho}(K)$ is the normal closure of $(xy)^2.L$, so it is isomorphic to the subgroup of $PSL(2, \mathbb{L})$ generated by

$$(XY)^2 = -I + (\lambda - \alpha)(XY)$$

together with its conjugates by elements of $\hat{\rho}(G)$. Let $Z = \phi(XY) \in A_5 \subset SU(2)$ denote the matrix obtained from XY by substituting $\lambda = \alpha$. Note that $\text{tr}(Z) = 0$, in other words, $Z \in sl_2(\mathbb{C})$. Since $(\lambda - \alpha)^2 = 0$ in \mathbb{L} , we also have

$$(XY)^2 = -I + (\lambda - \alpha)Z.$$

For similar reasons, for any $M \in \widehat{\rho}(G)$ we have

$$M(XY)^2M^{-1} = -I + \phi(M)Z\phi(M)^{-1}.$$

Moreover, since $(\lambda - \alpha)^2 = 0$ in \mathbb{L} we have, for any $A, B \in sl_2(\mathbb{C})$,

$$(-I + (\lambda - \alpha)A)(-I + \lambda B) = I + (\lambda - \alpha)(A + B).$$

Thus $K/L \cong \rho(K)$ is isomorphic to the additive subgroup of $sl_2(\mathbb{C})$ generated by MZM^{-1} for all $M \in \widehat{A_5} \subset SU(2)$. There are precisely 30 such conjugates of Z ; geometrically they correspond to the midpoints of the edges of a regular icosahedron centred at the origin in \mathbb{R}^3 , where we identify $SU(2)$ with the 3-sphere of unit-norm quaternions, and \mathbb{R}^3 with the space of purely imaginary quaternions. As an abelian group, therefore, $K/L \cong \rho(K) \cong \mathbb{Z}^6$ by Lemma 2.3.

Now K/L is also an A_5 -module. Its structure as an A_5 -module does not need to concern us, but Lemma 2.3 gives us some information about its structure as a C -module and as a V -module. This in turn gives information on the structure of $\Delta := (\rho)^{-1}(C)$.

Specifically, $H_0(C, K/L) = H_0(\Delta/K, K/L) \cong \mathbb{Z}_2^4 \oplus \mathbb{Z}^2$. It follows from the 5-term exact sequence

$$H_2(\Delta/L) \rightarrow H_2(\Delta/K) \rightarrow H_0(\Delta/K, K/L) \rightarrow H_1(\Delta/L) \rightarrow H_1(\Delta/K) \rightarrow 0$$

and the fact that $\Delta/K \cong \mathbb{Z}_2$ that $H_1(\Delta/L)$ has torsion-free rank 2, and that the torsion subgroup of $H_1(\Delta/L)$ is a non-zero vector space over \mathbb{Z}_2 . Hence we can define $N_1 = [\Delta, \Delta].L$ and $N_2 \supset N_1$ such that N_2/N_1 is the torsion-subgroup of $\Delta/N_1 = H_1(\Delta/L)$. That $N_1 \triangleleft \rho^{-1}(V)$ follows from the fact that $[\Delta, \Delta]$ and L are both normal in $\rho^{-1}(V)$. That $N_2 \triangleleft \rho^{-1}(V)$ follows from the fact that N_2/N_1 is characteristic in Δ/N_1 .

Finally, since V/C acts on $\mathbb{Z}^2 \cong \Delta/N_2$ by the antipodal map, it follows that $\rho^{-1}(V)/N_2 \cong \mathbb{Z}^2 \rtimes_{(-1)} \mathbb{Z}_2$, as required.

3 Main results

Theorem 3.1 *Let $G = \langle x, y | x^3 = y^5 = W(x, y)^2 = 1 \rangle$. If the trace polynomial $\tau_W(\lambda)$ of W has a multiple root, then G contains a nonabelian free subgroup.*

Proof. We may assume that the root α is one of $0, 1, (\pm 1 + \sqrt{5})/2$, for otherwise the result is immediate from Lemma 2.1. Let $\rho : G \rightarrow A_5$ be the essential representation corresponding to α , let $c = \rho(W) \in A_5$, $C = \{1, c\} \subset A_5$ the subgroup generated by c , and $V = \{1, a, b, c\} \subset A_5$ its centraliser in A_5 .

Let $N_1 \triangleleft N_2 \triangleleft \rho^{-1}(V) < G$ be the subgroups promised by Lemma 2.4. Let $\Gamma = \rho^{-1}(C) < \rho^{-1}(V)$ be the unique index 2 subgroup such that $N_2 \subset \Gamma$ and $\Gamma/N_2 \cong \mathbb{Z}^2$. Then Γ has index 30 in G and contains no conjugate of x or of y .

Applying the Reidemeister-Scheier process to the presentation of G in the statement of the Theorem, we obtain a presentation of Γ of the form

$$\Gamma = \langle k_1, \dots, k_{31} | r_1, \dots, r_{30}, s_1^2, s_2^2 \rangle,$$

where r_1, \dots, r_{10} are rewrites of conjugates of x^3 ; r_{11}, \dots, r_{16} are rewrites of conjugates of y^5 ; and r_{17}, \dots, r_{30} and $s_1^2 = W^2, s_2^2 = \hat{a}W^2\hat{a}^{-1}$ are rewrites of conjugates of W^2 , with $\rho(\hat{a}) = a$ and so $s_1 = W, s_2 = \hat{a}W\hat{a}^{-1} \in \Gamma$.

Let K be the 2-complex model of this presentation, $F = \mathbb{Z}_2$, and $p : \overline{K} \rightarrow K$ the regular cover corresponding to the normal subgroup $N_2 \triangleleft \Gamma$. Let $L \subset K$ be the subcomplex obtained by omitting the 2-cells corresponding to the relators s_1^2, s_2^2 , and let $\overline{L} := p^{-1}(L) \subset \overline{K}$.

Now, since Γ/N_2 is torsion-free, and since $s_1^2 = 1 = s_2^2$ in Γ , $s_1, s_2 \in N_2$. Hence each lift of each 2-cell s_i^2 ($i = 1, 2$) to \overline{K} is bounded by the square of some path in $\overline{K}^{(1)}$. As a consequence, the 2-cells in $\overline{K} \setminus \overline{L}$ represent elements of $H_2(\overline{K}, F)$, and it follows that the inclusion-induced map $H_1(\overline{L}, F) \rightarrow H_1(\overline{K}, F)$ is an isomorphism.

Since N_2/N_1 is a nonzero F -vector space, we have

$$H_1(\overline{L}, F) \cong H_1(\overline{K}, F) = H_1(N_2, F) \neq 0.$$

If $H_2(\overline{L}, F) = 0$, then by Lemma 2.2 it follows that $\dim_F H_1(N_2, F) = \infty$. On the other hand, if $H_2(\overline{L}, F) \neq 0$ then $H_2(\overline{L}, F)$ contains a free $F(\Gamma/N_2)$ -module of rank $> 0 = \chi(L)$, since $F(\Gamma/N_2)$ is an integral domain. In this case $H_1(\overline{L}, F)$ contains a non-zero free $F(\Gamma/N_2)$ -submodule, by [11, Proposition 2.1 and Theorem 2.2]. Again we deduce that $\dim_F H_1(N_2, F) = \infty$.

Thus the Bieri-Strebel invariant Σ of the $F(\Gamma/N_2)$ -module N_2/N_1 is a proper subset of S^1 [7, Theorem 2.4]. But by Lemma 2.3 (3) it follows that Σ is invariant under the antipodal map: $\Sigma = -\Sigma$. Hence $\Sigma \cup -\Sigma \neq S^1$, and it follows [7, Theorem 4.1] that Γ contains a nonabelian free subgroup, as claimed.

Corollary 3.2 (Main Theorem) *Let G be a generalised triangle group of type $(3, 5, 2)$. Then either G is virtually soluble or G contains a nonabelian free subgroup.*

Proof. By Theorem 3.1 and Lemma 2.1 the result follows unless $\tau_W(l)$ has only simple roots in the set $\{0, 1, (1 + \sqrt{5})/2, (-1 + \sqrt{5})/2\}$, in which case the degree k of $\tau_W(l)$ is at most equal to 4.

But the Rosenberger Conjecture is known for $k \leq 4$ [15].

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